

Borel chromatic number of closed graphs

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Abstract. We construct, for each countable ordinal ξ , a closed graph with Borel chromatic number two and Baire class ξ chromatic number \aleph_0 .

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1 Introduction

The study of the Borel chromatic number of analytic graphs on Polish spaces was initiated in [K-S-T]. In particular, the authors prove in this paper that the Borel chromatic number of the graph generated by a partial Borel function has to be in $\{1, 2, 3, \aleph_0\}$. They also provide a minimum graph \mathcal{G}_0 of uncountable Borel chromatic number. This last result had a lot of developments. For example, B. Miller gave in [Mi] some other versions of it, which helped him to generalize a number of known dichotomy theorems in descriptive set theory. The first author generalized in [L2] the \mathcal{G}_0 -dichotomy to any dimension making sense in classical descriptive set theory, and also used versions of \mathcal{G}_0 to study the non-potentially closed subsets of a product of two Polish spaces (see [L1]).

A study of the Δ_ξ^0 chromatic number of analytic graphs on Polish spaces was initiated in [L-Z1] and was motivated by the \mathcal{G}_0 -dichotomy. More precisely, let B be a Borel binary relation, on a Polish space X , having a Borel countable coloring (i.e., a Borel map $c : X \rightarrow \omega$ such that $c(x) \neq c(y)$ if $(x, y) \in B$). Is there a relation between the Borel class of B and that of the coloring? In other words, is there a map $k : \omega_1 \setminus \{0\} \rightarrow \omega_1 \setminus \{0\}$ such that any Π_ξ^0 binary relation having a Borel countable coloring has in fact a $\Delta_{k(\xi)}^0$ -measurable countable coloring, for each $\xi \in \omega_1 \setminus \{0\}$?

In [L-Z2], the authors give a negative answer: for each countable ordinal $\xi \geq 1$, there is a partial injection with disjoint domain and range $i : \omega^\omega \rightarrow \omega^\omega$, whose graph

- is $D_2(\Pi_1^0)$ (i.e., the difference of two closed sets),
- has Borel chromatic number two,
- has no Δ_ξ^0 -measurable countable coloring.

On the other hand, they note that an open binary relation having a finite coloring c has also a Δ_2^0 -measurable finite coloring (consider the differences of the $\overline{c^{-1}(\{n\})}$'s, for n in the range of the coloring). Note that an irreflexive closed binary relation on a zero-dimensional space has a continuous countable coloring (this coloring is Δ_2^0 -measurable in non zero-dimensional spaces). So they wonder whether we can build, for each countable ordinal $\xi \geq 1$, a closed binary relation with a Borel finite coloring but no Δ_ξ^0 -measurable finite coloring. This is indeed the case:

Theorem *Let $\xi \geq 1$ be a countable ordinal. Then there exists a partial injection with disjoint domain and range $f : \omega^\omega \rightarrow \omega^\omega$ whose graph is closed (and thus has Borel chromatic number two), and has no Δ_ξ^0 -measurable finite coloring (and thus has Δ_ξ^0 chromatic number \aleph_0).*

The previous discussion shows that this result is optimal. Its proof uses, among other things, the method used in [L-Z2] improving Theorem 4 in [M]. This method relates topological complexity and Baire category.

2 Mátrai sets

Before proving our main result, we recall some material from [L-Z2].

Notation. The symbol τ denotes the usual product topology on the Baire space ω^ω .

Definition 2.1 We say that a partial map $f : \omega^\omega \rightarrow \omega^\omega$ is **nice** if its graph $\text{Gr}(f)$ is a $(\tau \times \tau)$ -closed subset of $\omega^\omega \times \omega^\omega$.

The construction of P_ξ and τ_ξ , and the verification of the properties (1)-(3) from the next lemma (a corollary of Lemma 2.6 in [L-Z2]), can be found in [M], up to minor modifications.

Lemma 2.2 Let $1 \leq \xi < \omega_1$. Then there are $P_\xi \subseteq \omega^\omega$, and a topology τ_ξ on ω^ω such that

- (1) τ_ξ is zero-dimensional perfect Polish and $\tau \subseteq \tau_\xi \subseteq \Sigma_\xi^0(\tau)$,
- (2) P_ξ is a nonempty τ_ξ -closed nowhere dense set,
- (3) if $S \in \Sigma_\xi^0(\omega^\omega, \tau)$ is τ_ξ -nonmeager in P_ξ , then S is τ_ξ -nonmeager in ω^ω ,
- (4) if V, W are nonempty τ_ξ -open subsets of ω^ω , then we can find a τ_ξ -dense G_δ subset H of $V \setminus P_\xi$, a τ_ξ -dense G_δ subset L of $W \setminus P_\xi$, and a nice (τ_ξ, τ_ξ) -homeomorphism from H onto L .

The following lemma (a corollary of Lemma 2.7 in [L-Z2]) is a consequence of the previous one. It provides, among other things, a topology T_ξ that we will use in the sequel.

Lemma 2.3 Let $1 \leq \xi < \omega_1$. Then there is a disjoint countable family \mathcal{G}_ξ of subsets of ω^ω and a topology T_ξ on ω^ω such that

- (a) T_ξ is zero-dimensional perfect Polish and $\tau \subseteq T_\xi \subseteq \Sigma_\xi^0(\tau)$,
- (b) for any nonempty T_ξ -open sets V, V' , there are disjoint $G, G' \in \mathcal{G}_\xi$ with $G \subseteq V$, $G' \subseteq V'$, and there is a nice (T_ξ, T_ξ) -homeomorphism from G onto G' ,
and, for every $G \in \mathcal{G}_\xi$,
- (c) G is nonempty, T_ξ -nowhere dense, and in $\Pi_2^0(T_\xi)$,
- (d) if $S \in \Sigma_\xi^0(\omega^\omega, \tau)$ is T_ξ -nonmeager in G , then S is T_ξ -nonmeager in ω^ω .

The construction of \mathcal{G}_ξ and T_ξ ensures that T_ξ is $(\tau_\xi)^\omega$, where τ_ξ is as in Lemma 2.2. This topology is on $(\omega^\omega)^\omega$, identified with ω^ω . We will need the following consequence of the construction of \mathcal{G}_ξ and T_ξ .

Lemma 2.4 Let $1 \leq \xi < \omega_1$, and V be a nonempty T_ξ -open set. Then \overline{V}^τ is not τ -compact.

Proof. The fact that T_ξ is $(\tau_\xi)^\omega$ gives a finite sequence U_0, \dots, U_n of nonempty open subsets of $(\omega^\omega, \tau_\xi)$ with $U_0 \times \dots \times U_n \times (\omega^\omega)^\omega \subseteq V$. Thus \overline{V}^τ contains the τ -closed set $\overline{U_0}^\tau \times \dots \times \overline{U_n}^\tau \times (\omega^\omega)^\omega$, and it is enough to see that this last set is not τ -compact. This comes from the fact that the Baire space (ω^ω, τ) is not compact. \square

3 Proof of the main result

Before proving our main result, we give an example giving the flavour of the sequel. In [Za], the author gives a Hurewicz-like test to see when two disjoint subsets A, B of a product $Y \times Z$ of Polish spaces can be separated by an open rectangle. We set $\mathbb{A} := \{(n^\infty, n^\infty) \mid n \in \omega\}$,

$$\mathbb{B}_0 := \{(0^{m+1}(n+1)^\infty, (m+1)^{n+1}0^\infty) \mid m, n \in \omega\}$$

and $\mathbb{B}_1 := \{((m+1)^{n+1}0^\infty, 0^{m+1}(n+1)^\infty) \mid m, n \in \omega\}$. Then A is not separable from B by an open rectangle exactly when there are $\varepsilon \in 2$ and continuous maps $g : \omega^\omega \rightarrow Y$, $h : \omega^\omega \rightarrow Z$ such that $\mathbb{A} \subseteq (g \times h)^{-1}(A)$ and $\mathbb{B}_\varepsilon \subseteq (g \times h)^{-1}(B)$.

Example. Here we are looking for closed graphs with Borel chromatic number two and of arbitrarily high finite Δ_ξ^0 chromatic number n . There is an example with $\xi = 1$ and $n = 3$ where \mathbb{B}_0 is involved. We set $C := \{(2m)^\infty, (2m+1)^\infty \mid m \in \omega\} \cup \mathbb{B}_0$,

$$D := \{(2m)^\infty \mid m \in \omega\} \cup \{0^{m+1}(n+1)^\infty \mid m, n \in \omega\},$$

$$R := \{(2m+1)^\infty \mid m \in \omega\} \cup \{(m+1)^{n+1}0^\infty \mid m, n \in \omega\},$$

$$f((2m)^\infty) := (2m+1)^\infty \text{ and } f(0^{m+1}(n+1)^\infty) := (m+1)^{n+1}0^\infty.$$

This defines $f : D \rightarrow R$ whose graph is C . The first part of C is discrete, and thus closed. Assume that $(\alpha_k, \beta_k) := (0^{m_k+1}(n_k+1)^\infty, (m_k+1)^{n_k+1}0^\infty) \in \mathbb{B}_0$ and converges to $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ as k goes to infinity. We may assume that (m_k) is constant, and (n_k) too, so that $(\alpha, \beta) \in \mathbb{B}_0$, which is therefore closed. This shows that C is closed. Note that D, R are disjoint and Borel, so that C has Borel chromatic number two. Let Δ be a clopen subset of ω^ω . Let us prove that $C \cap \Delta^2$ or $C \cap (-\Delta)^2$ is not empty. We argue by contradiction. Then Δ or $\neg\Delta$ has to contain 0^∞ . Assume that it is Δ , the other case being similar. Then $0^{m+1}(n+1)^\infty \in \Delta$ if m is big enough. Thus $(m+1)^{n+1}0^\infty \notin \Delta$ if m is big enough. Therefore $(m+1)^\infty \notin \Delta$ if m is big enough. Thus $((2m)^\infty, (2m+1)^\infty) \in C \cap (-\Delta)^2$ if m is big enough, which is absurd.

We now turn to the general case. Our main lemma is as follows. We equip ω^m with the discrete topology τ_d , for each $m > 0$.

Lemma *Let $\xi \geq 1$ be a countable ordinal, $n \geq 1$ be a natural number, and $X := \omega \times \omega^\omega$. Then we can find a partial injection $f : X \rightarrow X$ and a disjoint countable family \mathcal{F} of subsets of X such that*

- (a) *f has disjoint domain and range,*
- (b) *$\text{Gr}(f)$ is $((\tau_d \times \tau) \times (\tau_d \times \tau))$ -closed,*
- (c) *there is no sequence $(\Delta_i)_{i < n}$ of Δ_ξ^0 subsets of $(X, \tau_d \times \tau)$ such that*
 - (i) $\forall i < n \quad \text{Gr}(f) \cap \Delta_i^2 = \emptyset$,
 - (ii) $\bigcup_{i < n} \Delta_i$ *is $(\tau_d \times T_\xi)$ -comeager in X ,*
- (d) *\mathcal{F} has the properties (b)-(d) in Lemma 2.3, where \mathcal{G}_ξ , ω^ω , T_ξ and τ are respectively replaced with \mathcal{F} , X , $\tau_d \times T_\xi$ and $\tau_d \times \tau$,*
- (e) $(\bigcup \mathcal{F}) \cap (\text{Domain}(f) \cup \text{Range}(f)) = \emptyset$.

Proof. We argue by induction on n .

The basic case $n = 1$

Let \mathcal{G}_ξ be the family given by Lemma 2.3. We split \mathcal{G}_ξ into two disjoint subfamilies \mathcal{G}_ξ^0 and \mathcal{G}_ξ^1 having the property (b) in Lemma 2.3. This is possible since the elements of \mathcal{G}_ξ are T_ξ -nowhere dense. Let $G_0, G_1 \in \mathcal{G}_\xi^0$ be disjoint, and φ be a nice (T_ξ, T_ξ) -homeomorphism from G_0 onto G_1 . We then set $f(0, \alpha) := (0, \varphi(\alpha))$ if $\alpha \in G_0$, and $\mathcal{F} := \{\{n\} \times G \mid n \in \omega \wedge G \in \mathcal{G}_\xi^1\}$. It remains to check that the property (c) is satisfied. We argue by contradiction, which gives $\Delta_0 \in \Delta_\xi^0$. By property (d) in Lemma 2.3, $\Delta_0 \cap (\{0\} \times G_\varepsilon)$ is $(\tau_d \times T_\xi)$ -comeager in $\{0\} \times G_\varepsilon$ for each $\varepsilon \in 2$. As f is a $(\tau_d \times T_\xi, \tau_d \times T_\xi)$ -homeomorphism, $\Delta_0 \cap (\{0\} \times G_0) \cap f^{-1}(\Delta_0 \cap (\{0\} \times G_1))$ is $(\tau_d \times T_\xi)$ -comeager in $\{0\} \times G_0$, which contradicts the fact that $\text{Gr}(f) \cap \Delta_0^2 = \emptyset$.

The induction step from n to $n+1$

The induction assumption gives f and \mathcal{F} . Here again, we split \mathcal{F} into two disjoint subfamilies \mathcal{F}^0 and \mathcal{F}^1 having the property (b) in Lemma 2.3, where $\mathcal{G}_\xi, \omega^\omega, T_\xi$ and τ are respectively replaced with $\mathcal{F}, X, \tau_d \times T_\xi$ and $\tau_d \times \tau$. Let (V_p) be a basis for the topology $\tau_d \times T_\xi$ made of nonempty sets. Fix $p \in \omega$. By Lemma 2.4, there is a countable family $(W_q^p)_{q \in \omega}$, with $(\tau_d \times \tau)$ -closed union, and made of pairwise disjoint $(\tau_d \times \tau)$ -clopen subsets of X intersecting V_p .

• Let $b : \omega \rightarrow \omega^2$ be a bijection. We construct, for $\vec{v} = (p, q) \in \omega^2$ and $\varepsilon \in 2$, and by induction on $b^{-1}(\vec{v})$,

- $G_\varepsilon^{\vec{v}} \in \mathcal{F}^0$,
- a nice $(\tau_d \times T_\xi, \tau_d \times T_\xi)$ -homeomorphism $\varphi^{\vec{v}} : G_0^{\vec{v}} \rightarrow G_1^{\vec{v}}$.

We want these objects to satisfy the following:

- $G_0^{\vec{v}} \subseteq (V_p \cap W_q^p) \setminus (\bigcup_{m < b^{-1}(\vec{v})} \overline{G_0^{b(m)} \cup G_1^{b(m)}}^{\tau_d \times T_\xi})$,
- $G_1^{\vec{v}} \subseteq V_q \setminus (G_0^{\vec{v}} \cup \bigcup_{m < b^{-1}(\vec{v})} \overline{G_0^{b(m)} \cup G_1^{b(m)}}^{\tau_d \times T_\xi})$.

• We now define the desired partial map $\tilde{f} : \omega \times \omega \times \omega^\omega \rightarrow \omega \times \omega \times \omega^\omega$, as well as $\tilde{\mathcal{F}} \subseteq 2^{\omega \times \omega \times \omega^\omega}$, as follows:

$$\tilde{f}(l, x) := \begin{cases} (p+1, \varphi^{p,q}(x)) & \text{if } l=0 \wedge x \in G_0^{p,q}, \\ (l, f(x)) & \text{if } l>0 \wedge x \in \text{Domain}(f). \end{cases}$$

and $\tilde{\mathcal{F}} := \{\{l\} \times G \mid l \in \omega \wedge G \in \mathcal{F}^1\}$. Note that \tilde{f} is well-defined and injective, by disjointness of the $(G_0^{\vec{v}} \cup G_1^{\vec{v}})$'s. Identifying X with $\omega \times \omega \times \omega^\omega$, we can consider \tilde{f} as a partial map from X into itself and $\tilde{\mathcal{F}}$ as a family of subsets of X (this identification is based on the identification of ω with $\omega \times \omega$).

(a), (d) and (e) are clearly satisfied.

(b) Assume that $((l_k, x_k), (m_k, y_k)) \in \text{Gr}(\tilde{f})$ tends to $((l, x), (m, y)) \in (\omega \times X)^2$ as k goes to infinity. We may assume that (l_k) and (m_k) are constant.

If $l=0$, then there is p such that $p+1=m$ and $(x_k, y_k) \in G_0^{p,q_k} \times G_1^{p,q_k}$. As $G_0^{p,q_k} \subseteq W_{q_k}^p$, we may also assume that (q_k) is also constant and equals q . As $\varphi^{p,q}$ is nice, $((l, x), (m, y)) \in \text{Gr}(f)$.

If $l>0$, then $(x_k, y_k) \in \text{Gr}(f)$. As $\text{Gr}(f)$ is $((\tau_d \times \tau) \times (\tau_d \times \tau))$ -closed, $((l, x), (m, y)) \in \text{Gr}(\tilde{f})$.

(c) We argue by contradiction, which gives $(\Delta_i)_{i \leq n}$. We may assume, without loss of generality, that $(\{0\} \times \omega \times \omega^\omega) \cap \Delta_n$ is not meager in $(\{0\} \times \omega \times \omega^\omega, \tau_d \times T_\xi)$. This gives $p \in \omega$ such that $(\{0\} \times V_p) \cap \Delta_n$ is $(\tau_d \times T_\xi)$ -comeager in $V_p' := \{0\} \times V_p$. As $V_p' \setminus \Delta_n \in \Sigma_\xi^0(\tau_d \times \tau)$, $(\{0\} \times G_0^{p,q}) \cap \Delta_n$ is $(\tau_d \times T_\xi)$ -comeager in $\{0\} \times G_0^{p,q}$ for each $q \in \omega$.

As $\text{Gr}(\tilde{f}) \cap \Delta_n^2 = \emptyset$ and the $\varphi^{\vec{v}}$'s are $(\tau_d \times T_\xi, \tau_d \times T_\xi)$ -homeomorphisms, $(\{p+1\} \times G_1^{p,q}) \cap \Delta_n$ is $(\tau_d \times T_\xi)$ -meager in $\{p+1\} \times G_1^{p,q}$, for each q .

As $(\omega \times \omega \times \omega^\omega) \setminus (\bigcup_{i \leq n} \Delta_i)$ is $(\tau_d \times T_\xi)$ -meager in $\omega \times \omega \times \omega^\omega$ and $\Delta_\xi^0(\tau_d \times \tau)$,

$$(\{p+1\} \times G_1^{p,q}) \setminus (\bigcup_{i \leq n} \Delta_i)$$

is $(\tau_d \times T_\xi)$ -meager in $\{p+1\} \times G_1^{p,q}$, for each q . Thus $(\{p+1\} \times G_1^{p,q}) \cap (\bigcup_{i < n} \Delta_i)$ is $(\tau_d \times T_\xi)$ -comeager in $\{p+1\} \times G_1^{p,q}$, for each q .

Claim *The set $(\{p+1\} \times \omega \times \omega^\omega) \cap (\bigcup_{i < n} \Delta_i)$ is $(\tau_d \times T_\xi)$ -comeager in $\{p+1\} \times \omega \times \omega^\omega$.*

Indeed, we argue by contradiction. This gives $W \in (\tau_d \times T_\xi) \setminus \{\emptyset\}$ such that

$$(\{p+1\} \times W) \cap (\bigcup_{i < n} \Delta_i)$$

is $(\tau_d \times T_\xi)$ -meager in $W' := \{p+1\} \times W$. Let $q \in \omega$ be such that $V_q \subseteq W$. Then $G_1^{p,q} \subseteq W$ and $\{p+1\} \times G_1^{p,q} \subseteq W'$. As $W' \cap (\bigcup_{i < n} \Delta_i) \in \Sigma_\xi^0(\tau_d \times \tau)$ and $(\{p+1\} \times G_1^{p,q}) \cap W' \cap (\bigcup_{i < n} \Delta_i)$ is $(\tau_d \times T_\xi)$ -comeager in $\{p+1\} \times G_1^{p,q}$, $W' \cap (\bigcup_{i < n} \Delta_i)$ is not $(\tau_d \times T_\xi)$ -meager in W' , which is absurd. \diamond

Now we set $\Delta'_i := (\{p+1\} \times \omega \times \omega^\omega) \cap \Delta_i$ if $i < n$. Note that $\Delta'_i \in \Delta_\xi^0(\{p+1\} \times \omega \times \omega^\omega, \tau_d \times \tau)$, $\text{Gr}(\tilde{f}) \cap (\Delta'_i)^2 = \emptyset$, and $\bigcup_{i < n} \Delta'_i$ is $(\tau_d \times T_\xi)$ -comeager in $\{p+1\} \times \omega \times \omega^\omega$, which contradicts the induction assumption. \square

In order to get our main result, it is enough to apply the main lemma to each $n \geq 1$. This gives $f_n : \omega \times \omega^\omega \rightarrow \omega \times \omega^\omega$. It remains to define $f : \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega) \rightarrow \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega)$ by $f(n, x) := f_n(x)$ (we identify $(\omega \setminus \{0\}) \times \omega \times \omega^\omega$ with ω^ω).

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